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Sliding Mode Control of Mechanical Systems with Bounded Disturbances via Output Feedback

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In this paper, a new sliding mode output feedback algorithm is developed to control the vibration of mechanical systems subjected to disturbance forces. It has been assumed that the number of states is greater than the number of sensors. The sliding mode control law is designed to be robust to external disturbances provided their upper bounds are known. A boundary layer has been introduced around each sliding hyperplane to eliminate the chattering phenomenon. The results from numerical simulations are presented to corroborate the validity of the proposed controller.

Introduction

THIS paper deals with the sliding mode control of a mechanical system described by a set of second-order differential equations with known bounds on external excitations. An example of such a system is a flexible rotor where the excitation is caused by the inevitable uncertainties in the rotor eccentricity. Although it is not possible to know the exact values of the rotor eccentricity, it is a quite straightforward task to estimate its maximum possible value. Using the sliding mode control theory,[1,2] a controller can be easily designed for robustness with respect to uncertainties in external disturbances, provided all the states are known. However, for flexible structures, which are infinite dimensional in theory, and other mechanical systems with a large number of states, it may not be practical or even possible to know all of the states. Consequently, it is important to develop the sliding mode controller based on a limited number of outputs, so that the robustness to external disturbances is guaranteed.

Sliding mode control has been used in the context of model reference adaptive control (MRAC) by various researchers.[3] In this approach, a complex system is decomposed into a series of interconnected subsystems. For each subsystem, a reference model is given and it is assumed that all states of the subsystems and the reference model are measurable. However, this approach requires the upper bounds of terms representing interactions among various subsystems that depend on all of the system states. These are not known a priori because the states are influenced by the controller. Furthermore, the presence of external disturbances was not considered. The problem of designing output feedback controllers via sliding mode control to stabilize multivariable plants has been investigated by Diong and Medanic[4] and Heck and Ferri.[5] However, they also do not consider the presence of external excitation, and their efforts are concentrated on the development of a stable sliding manifold and satisfaction of reaching conditions. The approach used by Diong and Medanic[4] is based on the simplex method, and they apply it to two types of dynamic controllers: a compensator type and an observer type. Heck and Ferri[5] described conditions to satisfy stability and reaching conditions. The stability condition is satisfied by pole placement techniques with output feedback. The reaching condition is satisfied through the selection of a matrix that also affects the time to reach the sliding surface. However, they did not present methods for determining this matrix. Recently, Yallapragada et al.[6] discussed a design method to obtain a controller that satisfies the reaching condition for variable structure controllers with output feedback. They also examined the robustness to external disturbances when the matching conditions are met. Wang and Fan[7] developed an interesting approach to design a sliding mode output feedback control. Their definition of sliding hyperplanes included an exponentially decaying term. Effectively, the system starts from being initially on the sliding hyperplanes. However, they did not address the issue of robustness to external disturbances. Kwan[8] extended the approach developed by Wang and Fan[7] by eliminating the exponentially decaying term and formulating a time-varying upper bound of states. He also examined the robustness to mismatched disturbances.[8] However, this disturbance vector has a special structure that may limit its applicability to many systems.

This paper develops a novel sliding mode output feedback control methodology for a mechanical system to guarantee robustness to bounded external excitation when the number of states is greater than the number of outputs. This robustness is achieved through the proper selection of gains associated with the nonlinear part of the control law and a suitable choice of estimated state dynamics. These gains are computed off-line and there is no restriction on the nature or structure of external excitation vector. Theoretical analysis of the stability of closed-loop system and estimated state error dynamics are presented when sensors and actuators are collocated. Using a two-degree-of-freedom spring-mass system (Fig. 1), numerical results are presented to illustrate the controller design procedure and the effectiveness of the controller as well.

Problem Statement

Consider a q-degree-of-freedom mechanical system described in physical coordinates by the following set of equations:

$$M_r\ddot{r} + G_r\dot{r} + K_r r = \bar{B}F_c + \bar{D}F_d(t) \tag{1}$$

Matrices M_r, G_r, and K_r represent the mass, gyroscopic/dissipation, and stiffness characteristics of the system, respectively. The $m \times 1 (m < q)$ vector F_c represents control forces, the $p \times 1$ vector $F_d(t)$ contains external disturbances and $\bar{B}$ and $\bar{D}$ are appropriately defined matrices. It is not assumed that the matching condition as described by Verghese et al.[9]; i.e., $\bar{D}F_d(t) = \bar{B}W_d(t)$, where $W_d(t)$ is a known bounded function, is met. The system of q second-order differential equations, Eq. (1) can be transformed into the state-space form

$$\dot{X} = AX + BF_c + DF_d(t) \tag{2}$$

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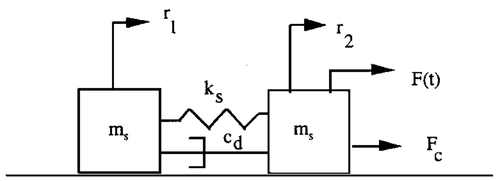

Fig. 1 Two-degree-of-freedom mechanical system.

where

$$X = \begin{bmatrix} r \\ \dot{r} \end{bmatrix}, \qquad A = \begin{bmatrix} 0 & I \\ -M_r^{-1}K_r & -M_r^{-1}G_r \end{bmatrix}$$
$$B = \begin{bmatrix} 0 \\ M_r^{-1}\bar{B} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ M_r^{-1}\bar{D} \end{bmatrix} \tag{3}$$

The measurements available for the controller design can be expressed in the output form as

$$Y = CX \tag{4}$$

where C is an appropriately dimensioned matrix. It is assumed that the displacement and velocity are measurable at a given location. If y_i and $\dot{y}_i$ denote the displacement and velocity at the ith location, respectively, and such measurements are available at m locations, the output vector will be $2m$ dimensional described as follows:

$$Y(t) = \begin{bmatrix} Y_1(t)^T & Y_2(t)^T \end{bmatrix}^T \tag{5}$$

where

$$Y_1(t) = [y_1, y_2, y_3, \ldots, y_m]^T \tag{6}$$

$$Y_2(t) = [\dot{y}_1, \dot{y}_2, \dot{y}_3, \ldots, \dot{y}_m]^T \tag{7}$$

Note that $\dot{Y}_1(t) = Y_2(t)$ and the dimension of the matrix C would be $2m \times n$ where $n = 2q$. It is assumed that $2m < n$ and that the bound on the magnitude of each element of the disturbance vector $F_d(t)$ is known. The objective is to design a sliding mode controller using output feedback only so that outputs are as close to desired values as possible.

Development of the Sliding Mode Control Algorithm

Let the error be defined as

$$\tilde{y}(t) = y(t) - y_d(t) \tag{8}$$

where $y(t)$ is the actual output and $y_d(t)$ is the desired response. For a vibration control problem, $y_d(t) = 0$; therefore, $\tilde{y}(t) = y(t)$. Following the approach by Asada and Slotine,[10] m sliding hyperplanes are defined as follows:

$$s_i(t) = \left(\frac{d}{dt} + \lambda_i\right)^2 \int_0^t y_i(\tau)\,d\tau$$
$$= \dot{y}_i(t) + 2\lambda_i y_i(t) + \lambda_i^2 \int_0^t y_i(\tau)\,d\tau, \qquad i = 1, 2, 3, \ldots, m \tag{9}$$

where λ_i is a positive number. Note that the number of sliding hyperplanes equals the number of control inputs. Using Eqs. (5–7) and (9),

$$S = PY + \Lambda_I Z \tag{10}$$

where

$$S = [s_1, s_2, s_3, \ldots, s_m]^T \tag{11}$$

$$P = [P_1 \quad P_2] \tag{12}$$

$$P_1 = \mathrm{diag}(2\lambda_1, 2\lambda_2, 2\lambda_3, \ldots, 2\lambda_m) \tag{13}$$

$$P_2 = \mathrm{diag}(1, 1, 1, \ldots, 1) \tag{14}$$

$$\Lambda_I = \mathrm{diag}\left(\lambda_1^2, \lambda_2^2, \lambda_3^2, \ldots, \lambda_m^2\right) \tag{15}$$

$$Z = \left[\int_0^t y_1(\tau)\,d\tau, \int_0^t y_2(\tau)\,d\tau, \int_0^t y_3(\tau)\,d\tau, \ldots, \int_0^t y_m(\tau)\,d\tau\right] \tag{16}$$

Differentiating Eq. (10) and noting that $\dot{Z} = Y_1$,

$$\dot{S} = P^* Y + \bar{P}\dot{Y} \tag{17}$$

where

$$P^* = [\Lambda_I \quad P_1] \tag{18}$$

and

$$\bar{P} = [0 \quad P_2] \tag{19}$$

Using Eqs. (2) and (4), Eq. (17) can be rewritten as

$$\dot{S} = P^*Y + U + \bar{P}CAX + \bar{P}CDF_d(t) \tag{20}$$

where

$$U = \bar{P}CBF_c \tag{21}$$

The best estimate of the controller or equivalent control corresponding to $\dot{S} = 0$ is

$$U_{\mathrm{eq}} = -P^*Y - \bar{P}CA\hat{X} \tag{22}$$

where $\hat{X}$ is the estimate of X. As will be seen later, the choice of $\hat{X}$ plays a key role in the off-line design of the sliding mode controller. A vector containing the sign function is added to the control input in Eq. (22) to satisfy the reaching condition

$$s_i(t)\dot{s}_i(t) \le -\eta_i |s_i(t)|, \qquad i = 1, 2, \ldots, m \tag{23}$$

where η_i is a positive real number. To eliminate the chattering phenomenon, the sign function is replaced by the saturation function. In essence, a boundary layer[10] is introduced around each sliding hyperplane so that the control input is a nonlinear function of the outputs outside the boundary layer and a linear function inside the boundary layer. The modified control input is taken to be

$$U = U_{\mathrm{eq}} - K\bar{S} \tag{24}$$

where K is the matrix of control gains associated with the nonlinear terms defined as

$$K = \mathrm{diag}(k_1, k_2, k_3, \ldots, k_m) \tag{25}$$

and

$$\bar{S} = [\mathrm{sat}(s_1/\phi_1), \mathrm{sat}(s_2/\phi_2), \ldots, \mathrm{sat}(s_m/\phi_m)] \tag{26}$$

The saturation function is defined as

$$\mathrm{sat}\left(\frac{s_i(t)}{\phi_i(t)}\right) = \begin{cases} \dfrac{s_i(t)}{\phi_i(t)} & \text{if} \quad \left|\dfrac{s_i(t)}{\phi_i(t)}\right| < 1 \\ \mathrm{sign}\left(\dfrac{s_i(t)}{\phi_i(t)}\right) & \text{otherwise} \end{cases} \tag{27}$$

The thickness of the boundary layer ϕ_i is determined by the balance condition developed by Asada and Slotine[10] as

$$\phi_i = k_i/\lambda_i \tag{28}$$

Substituting Eq. (24) into Eq. (2) yields the closed-loop system dynamics

$$\dot{X} = \tilde{A}X + \tilde{B}\bar{P}CA\hat{X} + DF_d + \tilde{B}K\bar{S} \tag{29}$$

where

$$\tilde{A} = A + \tilde{B}P^*C \tag{30}$$

$$\tilde{B} = -B(\bar{P}CB)^{-1} \tag{31}$$

Calculation of the Control Gain K

Substituting Eq. (24) into Eq. (20),

$$\dot{S} = \bar{P}CA(X - \hat{X}) - K\bar{S} + \bar{P}CDF_d(t) \qquad (32)$$

To satisfy the reaching condition Eq. (23),

$$K_v = [k_1, k_2, k_3, \ldots, k_m]^T \geq |\bar{P}CA||X - \hat{X}| + |\bar{P}CD||F_d(t)| + \eta \qquad (33)$$

where

$$\eta = [\eta_1, \eta_2, \eta_3, \ldots, \eta_m]^T \qquad (34)$$

The computation of elements of the gain vector K_v requires the upper bounds of the errors in the state estimate $(X - \hat{X})$ and the dynamic force created by the external disturbance. At the same time, $(X - \hat{X})$ will be, in general, dependent on the gain vector K_v. Hence, the dynamics of $\hat{X}$ are chosen such that the quantity $|X - \hat{X}|$ is independent of the gain vector K_v:

$$\dot{\hat{X}} = (\tilde{A} + \tilde{B}\bar{P}CA)\hat{X} + \tilde{B}K\bar{S}, \qquad \hat{X}(0) = \hat{X}_0 \qquad (35)$$

Subtracting Eq. (35) from Eq. (29) yields

$$\dot{X} - \dot{\hat{X}} = \tilde{A}(X - \hat{X}) + DF_d(t) \qquad (36)$$

The upper bound on the magnitude of the vector $X - \hat{X}$ can now be obtained as

$$|X - \hat{X}| \leq |e^{\tilde{A}t}||X_0 - \hat{X}_0| + \left|\int_0^t e^{\tilde{A}(t-\tau)} DF_d(\tau)\,d\tau\right| \qquad (37)$$

which is independent of K_v. Thus, for a given bound on the initial conditions of X, the upper bound on the magnitude $X - \hat{X}$ can be determined for any time t provided the bound of the disturbance force $F_d(t)$ is known. Since the choice of $\hat{X}(0)$ is arbitrary, it is selected to be zero. The detailed evaluation of the upper bound of $|X - \hat{X}|$ can be found in Lewis.[11]

Stability of Error Dynamics

To determine the stability of Eq. (36), which is used to determine the gain vector K_v, the location of the eigenvalues of $\tilde{A}$ should be checked. When sensors and actuators are collocated, it is shown next that all of these eigenvalues can be easily placed in the left half of the complex plane. Equation (36) can be written as a set of second-order differential equations as follows:

$$\ddot{r} - \ddot{\hat{r}} + A_1(\dot{r} - \dot{\hat{r}}) + A_2(r - \hat{r}) = M_r^{-1}\bar{D}F_d \qquad (38)$$

where

$$A_1 = M_r^{-1}G_r + M_r^{-1}\bar{B}\left(\bar{C}M_r^{-1}\bar{B}\right)^{-1}P_1\bar{C} \qquad (39)$$

$$A_2 = M_r^{-1}K_r + M_r^{-1}\bar{B}\left(\bar{C}M_r^{-1}\bar{B}\right)^{-1}\Lambda_I\bar{C} \qquad (40)$$

$$\bar{C} = \bar{B}^T \qquad (41)$$

The energy function E_s of the system can be expressed[12] as

$$E_s = \tfrac{1}{2}(\dot{r} - \dot{\hat{r}})^T(\dot{r} - \dot{\hat{r}}) + \tfrac{1}{2}(r - \hat{r})^T A_2(r - \hat{r}) \qquad (42)$$

Differentiating Eq. (42),

$$\dot{E}_s = (\dot{r} - \dot{\hat{r}})^T(\ddot{r} - \ddot{\hat{r}}) + (\dot{r} - \dot{\hat{r}})^T A_2(r - \hat{r}) \qquad (43)$$

where it is noted that the matrix A_2 is symmetric. Substituting Eq. (38) into Eq. (43) and neglecting the disturbance force, which does not affect the stability, gives

$$\dot{E}_s = -(\dot{r} - \dot{\hat{r}})^T A_1(\dot{r} - \dot{\hat{r}}) \qquad (44)$$

The Lyapunov's method of stability analysis[13] indicates that the system Eq. (38) is stable provided the matrices A_1 and A_2 are positive definite. It is assumed that the matrices $M_r^{-1}G_r$ and $M_r^{-1}K_r$ are either positive definite or positive semidefinite. Furthermore, second terms in Eqs. (39) and (40) are positive semidefinite. When $M_r^{-1}G_r$ and $M_r^{-1}K_r$ are positive definite, A_1 and A_2 are positive definite thereby insuring stability of Eq. (38). When $M_r^{-1}G_r$ and $M_r^{-1}K_r$ are positive semidefinite, A_1 and A_2 will be positive definite, provided eigenvectors corresponding to zero eigenvalues of both matrices on the right-hand sides of Eqs. (39) and (40) do not coincide.

Stability of Closed-Loop System

The closed-loop system dynamics inside boundary layers is obtained by combining Eqs. (26), (29), (32), and (35) as

$$\begin{bmatrix} \dot{X} \\ \dot{\hat{X}} \\ \dot{S} \end{bmatrix} = A_{\text{sys}} \begin{bmatrix} X \\ \hat{X} \\ S \end{bmatrix} + \begin{bmatrix} D \\ 0 \\ \bar{P}CD \end{bmatrix} F_d(t) \qquad (45)$$

where

$$A_{\text{sys}} = \begin{bmatrix} \tilde{A} & \tilde{B}\bar{P}CA & \tilde{B}K' \\ 0 & \tilde{A} + \tilde{B}\bar{P}CA & \tilde{B}K' \\ \bar{P}CA & -\bar{P}CA & -K' \end{bmatrix} \qquad (46)$$

$$K' = \operatorname{diag}(k_1/\phi_1, k_2/\phi_2, k_3/\phi_3, \ldots, k_m/\phi_m) \qquad (47)$$

For the stability of the closed-loop system, all eigenvalues of A_{sys} must be in the left half of the complex plane. As shown in the appendix,

$$\det(\alpha I_{2n+m} - A_{\text{sys}}) = \det(\alpha I_n - \tilde{A}) \times \det[\alpha I_n - (\tilde{A} + \tilde{B}\bar{P}CA)]\det(\alpha I_m + K') \qquad (48)$$

where det denotes the determinant. Since $-K'$ is guaranteed to be a stable matrix, the closed-loop system is stable, provided the eigenvalues of $\tilde{A}$ and $(\tilde{A} + \tilde{B}\bar{P}CA)$ are in the left half of the complex plane. The stability of $\tilde{A}$ has already been discussed in the preceding section. To examine the stability of $(\tilde{A} + \tilde{B}\bar{P}CA)$, the following system is considered:

$$\dot{w} = (\tilde{A} + \tilde{B}\bar{P}CA)w \qquad (49)$$

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \qquad (50)$$

Equation (49) is then converted to the following system of second-order differential equations:

$$\ddot{w}_1 + A_3\dot{w}_1 + A_4 w_1 = 0 \qquad (51)$$

where

$$A_3 = A_1 - M_r^{-1}\bar{B}\left(\bar{C}M_r^{-1}\bar{B}\right)^{-1}\bar{C}M_r^{-1}G_r \qquad (52)$$

$$A_4 = A_2 - M_r^{-1}\bar{B}\left(\bar{C}M_r^{-1}\bar{B}\right)^{-1}\bar{C}M_r^{-1}K_r \qquad (53)$$

The terms added to A_1 and A_2 in Eqs. (52) and (53) make matrices A_3 and A_4 asymmetric. In other words, Eq. (51) represents the dynamics of a system having asymmetric damping and stiffness matrices. Therefore, the Lyapunov function used earlier cannot be directly applied. Inman[14] discussed the extension of this approach to include systems with asymmetric damping and stiffness matrices by using the concept of symmetrizable matrices. If the matrix A_4 is symmetrizable, it can be written as a product of two symmetric matrices, one of which is positive definite; i.e.,

$$A_4 = T_1T_2, \qquad T_1 = T_1^T > 0, \quad T_2 = T_2^T \qquad (54)$$

Since the matrix T_1 is positive definite, it can be written as $T_1 = QQ^T$, where Q is nonsingular.[15] Using the similarity transformation $w_1 = Qh$, Eq. (51) can be rewritten as

$$\ddot{h} + Q^{-1}A_3Q\dot{h} + Q^TT_2Qh = 0 \qquad (55)$$

The system, Eq. (55), is stable if the symmetric part of $Q^{-1}A_3Q$ and the symmetric matrix Q^TT_2Q are positive definite.[15] It should be noted that this approach is based on a Lyapunov function and, therefore, conditions described here are only sufficient for stability.

Illustrative Example

To illustrate the control technique, consider the two-degree-of-freedom mechanical system depicted in Fig. 1. Here, it is assumed that only r_2 and $\dot{r}_2$ are available for measurement. The F_c is the control force and $F(t)$ is the disturbance force. When the disturbance force $F(t)$ is acting on the right mass of Fig. 1, the matching conditions are met. To simulate the nonmatching condition, $F(t)$ will be taken to be acting on the left mass. The equations of motion, when $F(t)$ is applied to the right mass, can be put into first-order form after Eq. (2), where

$$X = [r_1 \quad r_2 \quad \dot{r}_1 \quad \dot{r}_2]^T$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_s/m_s & k_s/m_s & -c_d/m_s & c_d/m_s \\ k_s/m_s & -k_s/m_s & c_d/m_s & -c_d/m_s \end{bmatrix} \tag{56}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_s \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_s \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{57}$$

Let the disturbance force be described by

$$F(t) = A_0 \cos(\Omega t) \tag{58}$$

where A_0 is the magnitude of the disturbance.

The parameters for the system model and control used in the simulation are given as $m_s = 5$ kg, $c_d = 5$ Ns/m, $k_s = 500$ N/m, $\Omega = 14.137$ rad/s, $\lambda = 25$ rad/s, and $\eta = 1$ N/kg. It is assumed that $|X_1(0)| \leq 0.3$ m, $|X_2(0)| \leq 0.2$ m, $|X_3(0)| \leq 0.1$ m/s, and $A_0 \leq 100$ N. The bound on $X_4(0)$ was taken to be zero. With these upper bounds, the upper bounds on $X - \hat{X}$ are determined from Eq. (37) and the gain from Eq. (33). For this example, the control u is scalar in nature; therefore, the number of sliding hyperplanes is one and there is only one gain k_1 to determine.

Matching Conditions Satisfied (Disturbance Force on the Right Mass)

From the simulation, the largest value of $(X - \hat{X})_i$ is determined and compared directly to the value obtained analytically from Eq. (37). In the simulation, the initial conditions are chosen to be $[0.3 \;\; 0.2 \;\; 0.0 \;\; 0.0]^T$ and $[0.0 \;\; 0.0 \;\; 0.0 \;\; 0.0]^T$ for X and $\hat{X}$, respectively. The analytical and simulated magnitudes of the upper bounds of $X - \hat{X}$ are as follows:

$$|X - \hat{X}|_{\text{analytical}} \leq [0.301 \quad 0.200 \quad 3.305 \quad 2.704]^T \tag{59}$$

$$|X - \hat{X}|_{\text{simulated}} \leq [0.300 \quad 0.200 \quad 2.219 \quad 1.381]^T \tag{60}$$

It can be seen that the analytical values are greater than or equal to all the simulated values. On the basis of analytically obtained upper bounds, Eq. (59), the gain k_1 is found to be 77.1 N/kg. Figure 2 shows the stable closed-loop time response for the output $y_1 = r_2$. Figure 3 shows a plot of $s(t)$ versus time. It is found that the system remains inside the boundary layer after about 0.13 s. To gain more insight into the closed-loop system dynamics, the poles and zeros of the closed-loop transfer function relating the output (r_2) to the disturbance force are determined and can be found in Table 1. Note that the poles of the system are the same as the eigenvalues of the matrix A_{sys}. It is observed that there are four pole/zero cancellations and cancelled poles are the eigenvalues of the matrix $(\tilde{A} + \tilde{B}\bar{P}CA)$, which are also stable. The pole–zero cancellation can be explained using the fact that the zero modes shapes can be

Table 1 Poles and zeros of closed-loop transfer function (matching conditions satisfied)

Poles	Zeros with respect to output $y = r_2$
$-0.50 + j9.99$	$-0.50 + j9.99$
$-0.50 - j9.99$	$-0.50 - j9.99$
-25.00	-25.00
-25.00	-25.00
$-0.82 + j9.39$	$-0.50 + j9.99$
$-0.82 - j9.39$	$-0.50 - j9.99$
$-25.18 + j8.34$	0.00
$-25.18 - j8.34$	
-25.00	

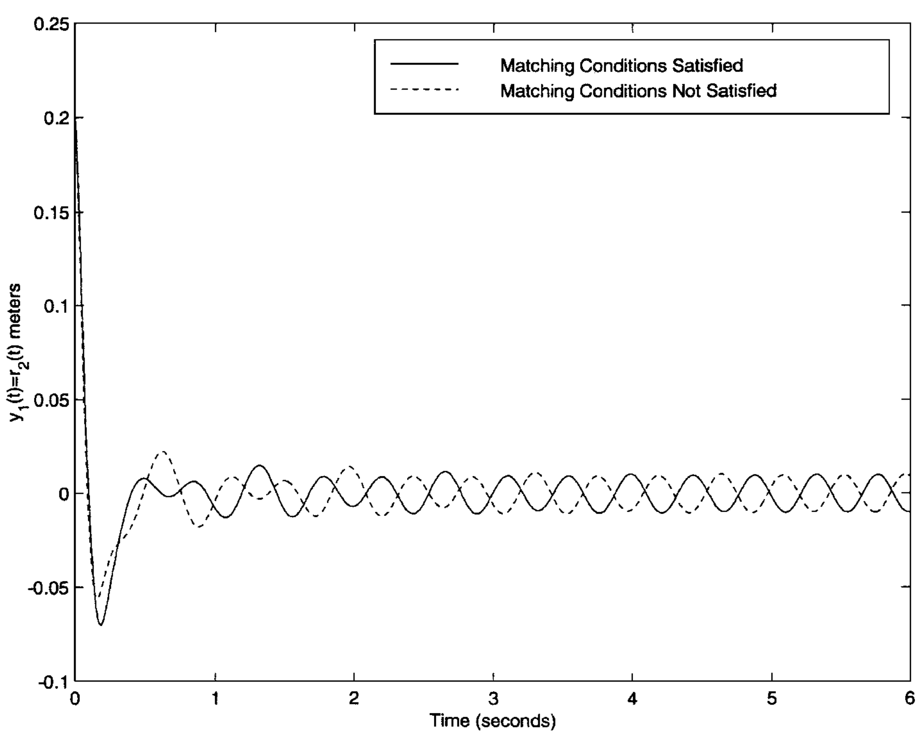

Fig. 2 Response of system for $\Omega = 14.137$ rad/s.

regarded as the solution of constrained modes problem.[16] When sliding mode control is designed, the output ($y = r_2$) is forced to be zero and a constrained structure is obtained. With $r_2 = 0$, two of the poles of the constrained structure can easily be obtained by the resulting single degree-of-freedom system as follows: $-0.50 \pm j9.99$. The poles at -25.00 correspond to the value of the parameter λ and the m eigenvalues of the $-K'$ matrix are equal to λ.

In the simulation results just given, it has been assumed that both position and velocity measurements are independently available for the implementation of the control law. In practice, this may not be the case because there might only be one type of sensor at each sensor station. For example, for a rotor/magnetic bearing setup,[11] only position sensors are available. Additional simulations, in which it is assumed that only position measurements are available, have been performed. The control law is implemented digitally with a sampling period of 0.001 s, and the velocities at sensor locations are obtained by difference equations. The results are very close to those of Figs. 2 and 3. In addition the observer as defined by Eq. (35) assumes a perfect knowledge of system matrices A, B, and D. To investigate the effects of parametric errors, a number of simulations have been performed in which the stiffness, mass, and damping parameters of the A matrix of Eq. (2) have been varied by $\pm 20\%$ from their nominal values used in the construction of the $\tilde{A} + \tilde{B}\bar{P}CA$ matrix of Eq. (35). The results are very close to those in Figs. 2 and 3. Future efforts will be aimed at developing a methodology to account theoretically for parametric errors in the plant and control matrices and unmodeled dynamics.

To illustrate the Lyapunov's stability analysis for the matrix $(\tilde{A} + \tilde{B}\bar{P}CA)$, matrices that appear in Eq. (55) are obtained. First, the A_4 matrix is found to be as follows:

$$A_4 = \begin{bmatrix} k_s/m_s & -k_s/m_s \\ 0 & \lambda^2 \end{bmatrix} \tag{61}$$

It can be shown that this matrix is symmetrizable if $\lambda \neq \sqrt{(k_s/m_s)}$. For the given values of λ and k_s, a combination of T_1 and T_2 as defined in Eq. (54) is

$$T_1 = \begin{bmatrix} 1.0376 & -0.1976 \\ -0.1976 & 1.0376 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 100 & 19.04 \\ 19.04 & 605.95 \end{bmatrix}$$

Lastly,

$$Q^{-1}A_3Q = \begin{bmatrix} 1.19 & -0.98 \\ -9.47 & 49.81 \end{bmatrix}$$
$$Q^TT_2Q = \begin{bmatrix} 119.04 & -98.15 \\ -98.15 & 605.96 \end{bmatrix} \tag{62}$$

It can be easily seen that the symmetric part of $Q^{-1}A_3Q$ and Q^TT_2Q are positive definite.

Matching Condition Not Satisfied (Disturbance Force on the Left Mass)

Figure 2 shows the response of the system when the disturbance force is located on the left mass. In this case, the D matrix is $[0 \;\; 0 \;\; 1/m_s \;\; 0]^T$ and the matching condition is violated. Again, the gain k_1 is calculated from Eq. (33) and is found to be 80.3 N/kg. Furthermore,

$$|X - \hat{X}|_{\text{analytical}} \leq [0.513 \quad 0.200 \quad 5.490 \quad 2.458]^T \tag{63}$$

$$|X - \hat{X}|_{\text{simulated}} \leq [0.453 \quad 0.200 \quad 5.161 \quad 1.621]^T \tag{64}$$

As expected, each element of the $|X - \hat{X}|_{\text{analytical}}$ vector is greater than or equal to its corresponding element in the $|X - \hat{X}|_{\text{simulated}}$ vector. Figures 2 and 3 show that the response is stable and system remains inside the boundary layer after 0.11 s. Again, there are pole/zero cancellations (Table 2) and the cancelled poles are the eigenvalues of the $(\tilde{A} + \tilde{B}\bar{P}CA)$ matrix. As with the case of

Table 2 Poles and zeros of closed-loop transfer function (matching condition not satisfied)

Poles	Zeros with respect to output $y = r_2$
$-0.50 + j9.99$	$-0.50 + j9.99$
$-0.50 - j9.99$	$-0.50 - j9.99$
-25.00	-25.00
-25.00	-25.00
$-0.82 + j9.39$	-100.00
$-0.82 - j9.39$	0.00
$-25.18 + j8.34$	
$-25.18 - j8.34$	
-25.00	

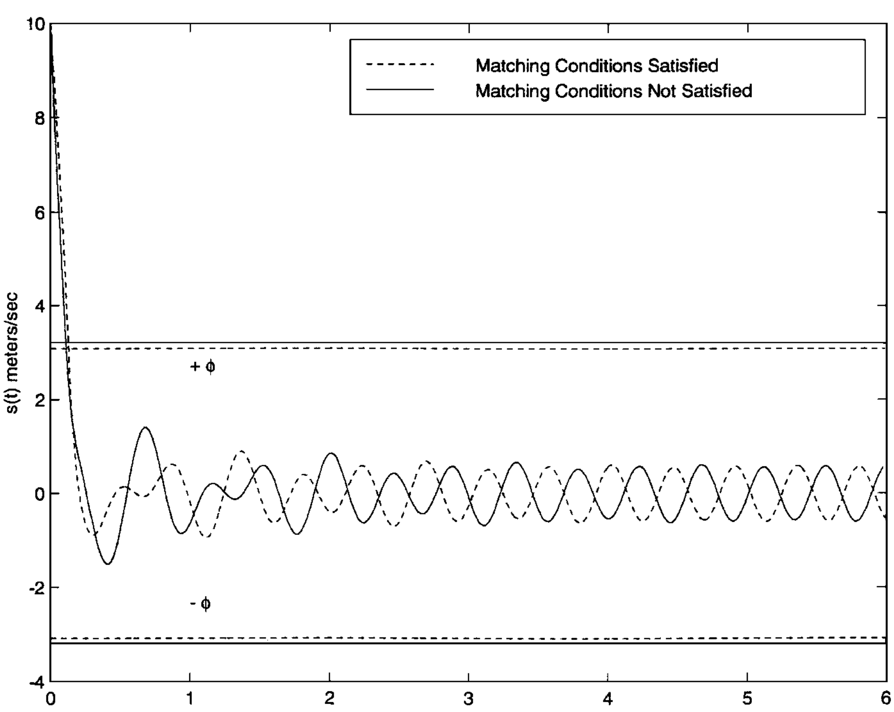

Fig. 3 $s(t)$ vs time for $\Omega = 14.137$ rad/s.

disturbance force on the right mass, simulation results show robust performance in the presence of parametric uncertainty and the use of only a position sensor, as well.

Conclusions

In this paper, a new sliding mode output feedback control algorithm has been developed for mechanical systems to guarantee robustness to bounded external disturbances. The robustness is achieved through the proper selection of the gain vector associated with the nonlinear part of the sliding mode control law. However, determining this gain vector is not a straightforward process when all of the states are not available. In this context, a novel and very systematic approach has been presented to determine this gain vector off-line by a suitable choice of estimated state dynamics. Furthermore, it is not necessary for disturbance forces to satisfy matching conditions. Numerical results for a two-degree-of-freedom mechanical system establish the validity of the new sliding mode control algorithm. It has also been observed that the sliding mode output feedback leads to a stable constrained structure, and consequently, stable pole/zero cancellations.

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Appendix: Eigenvalues of a Closed-Loop System

Lemma:

$$\det(\alpha I_{2n+m} - A_{\text{sys}}) = \det(\alpha I_n - \tilde{A}) \times \det[(\alpha I_n - (\tilde{A} + \tilde{B}\bar{P}CA)]\det(\alpha I_m + K') \tag{A1}$$

where A_{sys} is defined by Eq. (46).

Proof: Using elementary row operations,

$$\det(\alpha I_{2n+m} - A_{\text{sys}})$$

$$= \det\begin{bmatrix} \alpha I_n - \tilde{A} & -\tilde{B}\bar{P}CA & -\tilde{B}K' \\ 0 & \alpha I_n - (\tilde{A} + \tilde{B}\bar{P}CA) & -\tilde{B}K' \\ -\bar{P}CA & \bar{P}CA & \alpha I_m + K' \end{bmatrix}$$

$$= \det\begin{bmatrix} \alpha I_n - \tilde{A} & \tilde{A} - \alpha I_n & 0 \\ 0 & \alpha I_n - (\tilde{A} + \tilde{B}\bar{P}CA) & -\tilde{B}K' \\ -\bar{P}CA & \bar{P}CA & \alpha I_m + K' \end{bmatrix}$$

$$= \det\begin{bmatrix} \alpha I_m - \tilde{A} & 0 & 0 \\ 0 & \alpha I_n - (\tilde{A} + \tilde{B}\bar{P}CA) & -\tilde{B}K' \\ -\bar{P}CA & 0 & \alpha I_m + K' \end{bmatrix}$$

$$= \det\begin{bmatrix} \alpha I_m - \tilde{A} & 0 & 0 \\ 0 & \alpha I_n - (\tilde{A} + \tilde{B}\bar{P}CA) & 0 \\ 0 & 0 & \alpha I_m + K' \end{bmatrix}$$

$$= \det(\alpha I_n - \tilde{A})\det[\alpha I_n - (\tilde{A} + \tilde{B}\bar{P}CA)]\det(\alpha I_m + K')$$

This completes the proof.

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